INTEGRAL ('ANTI-DERIVATIVE') PROOF (BY RIEMANN SUMS)

Let f be continuous on the interval $[a, b]$, and let F be an antiderivative of f. Begin with the quantity

$$F(b) - F(a).$$

Let there be numbers

$x_1, \ldots, x_n$

such that

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

It follows that

$$F(b) - F(a) = F(x_n) - F(x_0).$$

Now, we add each $F(xi)$ along with its additive inverse, so that the resulting quantity is equal:

$$\begin{aligned} F(b) - F(a) &= F(x_n) + [-F(x_{n-1}) + F(x_{n-1})] + \ldots + [-F(x_1) + F(x_1)] - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) + \ldots - F(x_1)] + [F(x_1) - F(x_0)] \end{aligned}$$

The above quantity can be written as the following sum:

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \qquad (1)$$

Next we will employ the mean value theorem. Stated briefly,

Let F be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists some c in (a, b) such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}.$$

It follows that

$$F'(c)(b - a) = F(b) - F(a).$$

The function F is differentiable on the interval $[a, b]$; therefore, it is also differentiable and continuous on each interval x_{i-1}. Therefore, according to the mean value theorem (above),

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}).$$

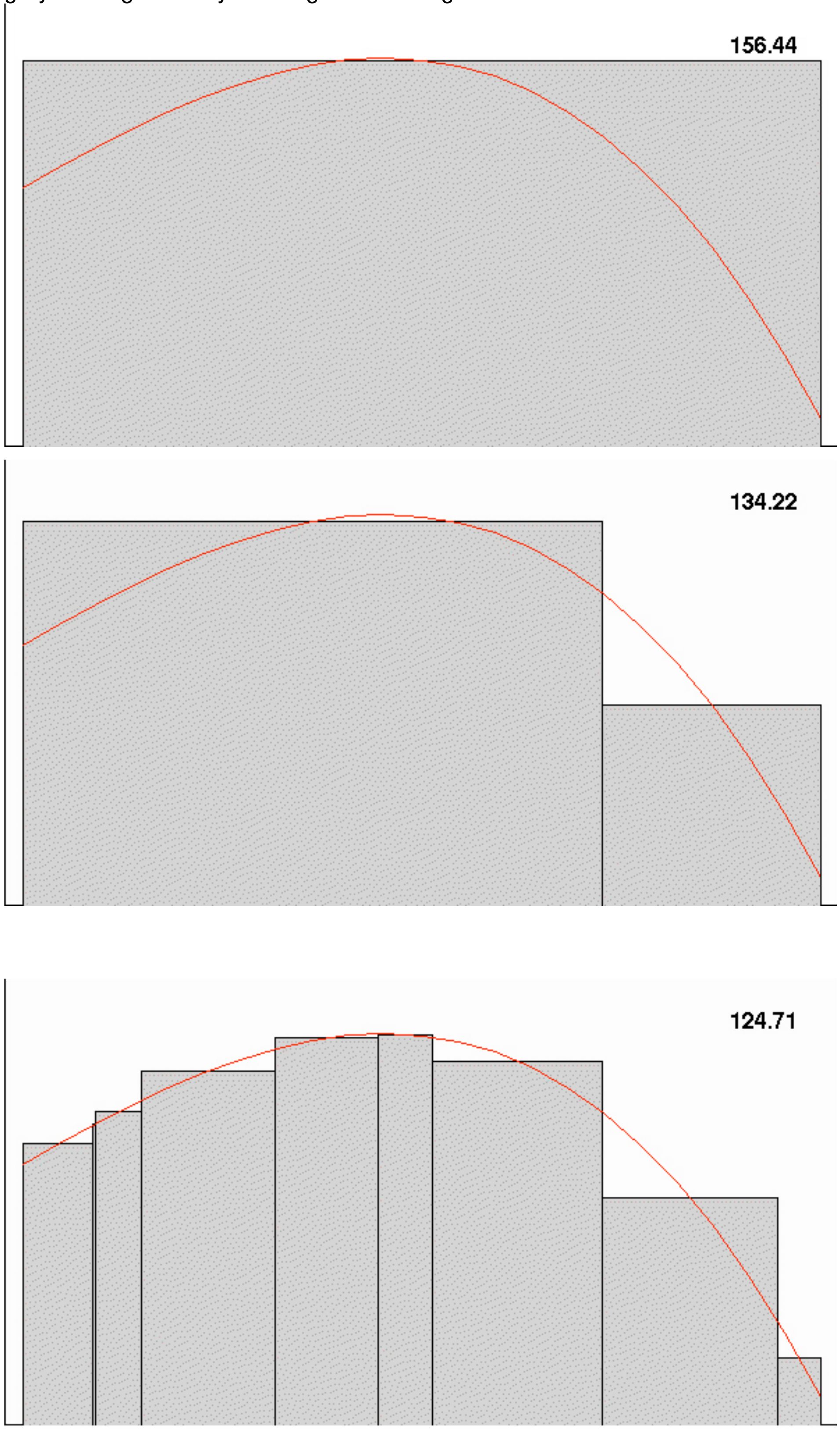
Substituting the above into (1), we get

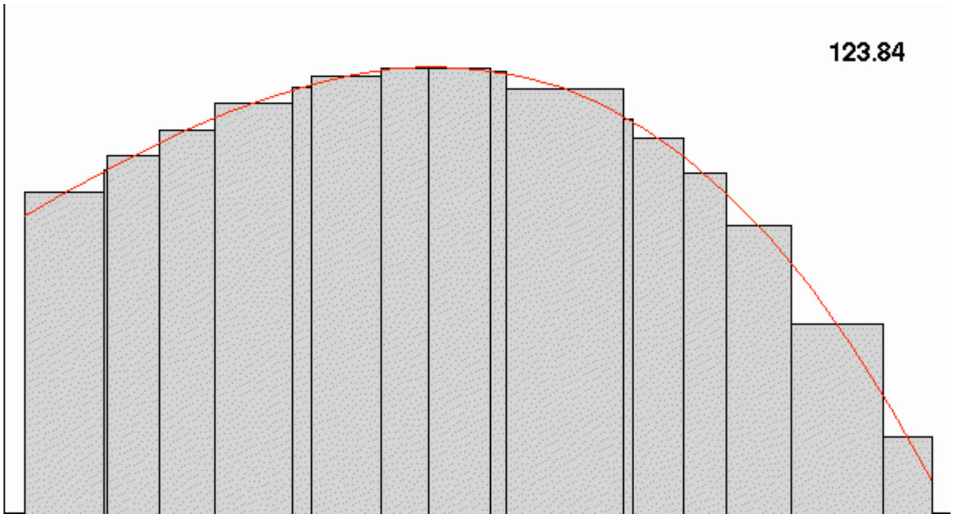
$$F(b) - F(a) = \sum_{i=1}^{n} [F'(c_i)(x_i - x_{i-1})].$$

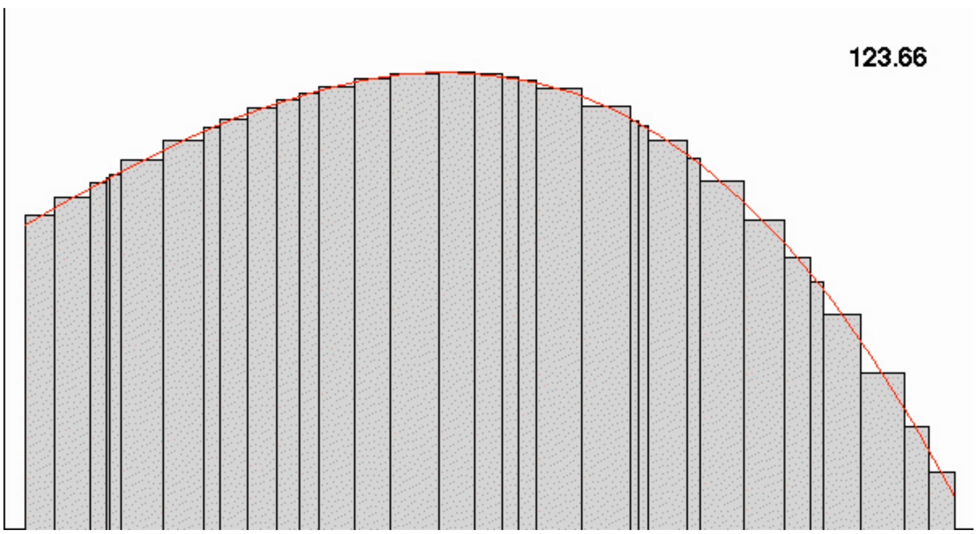
The assumption implies $F'(c_i) = f(c_i)$. Also, $x_i - x_{i-1}$ can be expressed as Δx of partition i.

$$F(b) - F(a) = \sum_{i=1}^{n} [f(c_i)(\Delta x_i)] \qquad (2)$$

A converging sequence of Riemann sums. The numbers in the upper right are the areas of the grey rectangles. They converge to the integral of the function.







Notice that we are describing the area of a rectangle, with the width times the height, and we are adding the areas together. Each rectangle, by virtue of the Mean Value Theorem, describes an approximation of the curve section it is drawn over. Also notice that Δx_i does not need to be the same for any value of i, or in other words that the width of the rectangles can differ. What we have to do is approximate the curve with n rectangles. Now, as the size of the partitions get smaller and n increases, resulting in more partitions to cover the space, we will get closer and closer to the actual area of the curve.

By taking the limit of the expression as the norm of the partitions approaches zero, we arrive at the Riemann integral. That is, we take the limit as the largest of the partitions approaches zero in size, so that all other partitions are smaller and the number of partitions approaches infinity.

So, we take the limit on both sides of (2). This gives us

$$\lim_{\|\Delta\|\to 0} F(b) - F(a) = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} [f(c_i)(\Delta x_i)] \, .$$

Neither $F(b)$ nor $F(a)$ is dependent on $\|\Delta\|$, so the limit on the left side remains $F(b)$ - $F(a)$.

$$F(b) - F(a) = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} [f(c_i)(\Delta x_i)]$$

The expression on the right side of the equation defines an integral over f from a to b. Therefore, we obtain

$$F(b) - F(a) = \int_a^b f(x)\, \mathrm{d}x$$

which completes the proof.